Embedded graphs

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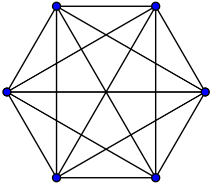
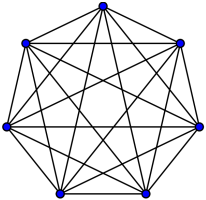
Exercise

Embed K_6 and K_7 into a torus.





2-cell embeddings of graphs

Definition

A 2-*cell embedding* of a graph G into a surface Σ is such an embedding, so that after you remove the graph, the resulting surface will be a disjoint union of discs.

Theorem

Suppose graph G is 2-cell embedded into a sphere with g handles. Then

$$V - E + F = 2 - 2g,$$

where V is the number of vertices, E is the number of edges and F is the number of faces.

Theorem

For any graph G there exists a number g so that G can be drawn on a sphere with g handles.

Definition

Define $\gamma(G)$ — minimal number g, so that G can be embedded into a sphere of genus g.

So $\gamma(K_5) = \gamma(K_{3,3}) = 1$, $\gamma(K_8) \geq 2$, $\gamma(\mathit{tree}) - ?$

Theorem

We have $\gamma(K_n) \geq \frac{(n-3)(n-4)}{12}$.
More generally, for polyhedral graphs, $\gamma(G) \geq 1 - \frac{V}{2} + \frac{E}{6}$.

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Corollary: $\gamma(K_8) \geq 2$.

Euler characteristic

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So far we know that it is correctly defined for spheres with handles.

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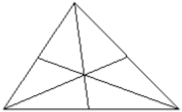

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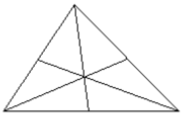

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Coarsening

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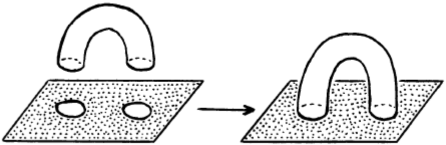
Exercise: compute Euler characteristic of $\mathbb{R}P^2, K^2, T^2, S^2$.

Attaching a Möbius strip

Attaching a handle:



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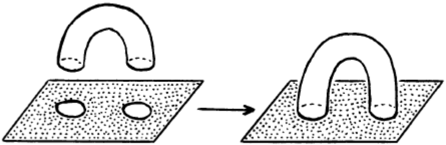
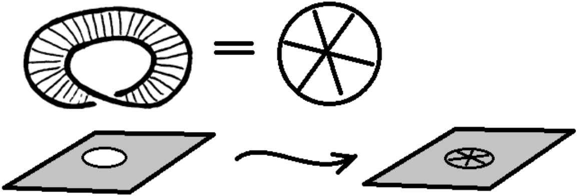


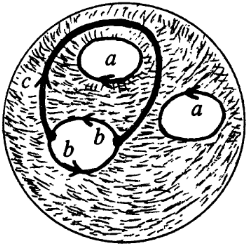
Attaching a Möbius band:



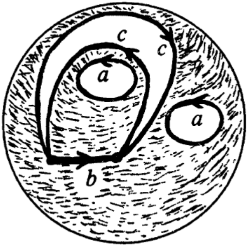
Lemma

Sphere with one handle and one Möbius band is homeomorphic to a sphere with 3 Möbius bands.

Proof:



(a)



(b)

Question: How does $\chi(\Sigma)$ change when attaching a handle? When attaching a Möbius strip?